Problem 4.58

An electron is in the spin state

$$\chi = A\begin{pmatrix} 1-2i \\ 2 \end{pmatrix}.$$

(a) Determine the constant A by normalizing χ.

(b) If you measured S_z on this electron, what values could you get, and what is the probability of each? What is the expectation value of S_z?

(c) If you measured S_x on this electron, what values could you get, and what is the probability of each? What is the expectation value of S_x?

(d) If you measured S_y on this electron, what values could you get, and what is the probability of each? What is the expectation value of S_y?

Solution

Part (a)

Determine A by requiring the spin state to be normalized.

$$\chi = \begin{bmatrix} A(1-2i) \\ 2A \end{bmatrix} \quad \Rightarrow \quad |A(1-2i)|^2 + |2A|^2 = 1$$

$$A^2|1-2i|^2 + 4A^2 = 1$$

$$A^2(1-2i)(1-2i)^* + 4A^2 = 1$$

$$A^2(1-2i)(1+2i) + 4A^2 = 1$$

$$A^2(1+4) + 4A^2 = 1$$

$$9A^2 = 1$$

$$A^2 = \frac{1}{9}$$

$$A = \pm\frac{1}{3}$$

Either sign works. Choose the plus sign.

$$A = \frac{1}{3}$$

Therefore, the normalized spin state is

$$\chi = \frac{1}{3}\begin{bmatrix} 1-2i \\ 2 \end{bmatrix}.$$

Part (b)

The matrix representing the z-component of the spin angular momentum S_z is

$$\mathsf{S}_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{bmatrix}.$$

Determine its eigenvalues.

$$\det(\mathsf{S}_z - \lambda \mathsf{I}) = 0$$

$$\begin{vmatrix} \frac{\hbar}{2} - \lambda & 0 \\ 0 & -\frac{\hbar}{2} - \lambda \end{vmatrix} = 0$$

$$\left(\frac{\hbar}{2} - \lambda\right)\left(-\frac{\hbar}{2} - \lambda\right) - (0)(0) = 0$$

$$\left(\lambda - \frac{\hbar}{2}\right)\left(\lambda + \frac{\hbar}{2}\right) = 0$$

$$\lambda = \pm\frac{\hbar}{2}$$

These are the possible values that can be obtained by measuring S_z. Now let

$$\lambda_- = -\frac{\hbar}{2} \quad \text{and} \quad \lambda_+ = +\frac{\hbar}{2},$$

and determine the corresponding eigenvectors (or rather eigenspinors in this context).

$$(\mathsf{S}_z - \lambda_- \mathsf{I})\chi_- = \mathsf{0} \qquad (\mathsf{S}_z - \lambda_+ \mathsf{I})\chi_+ = \mathsf{0}$$

$$\begin{bmatrix} \frac{\hbar}{2} + \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} + \frac{\hbar}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} \frac{\hbar}{2} - \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} - \frac{\hbar}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \hbar & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 \\ 0 & -\hbar \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\hbar x_1 = 0 \qquad -\hbar x_2 = 0$$

$$x_1 = 0 \qquad x_2 = 0$$

$$\chi_- = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} \qquad \chi_+ = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

Choose x_1 and x_2 so that the eigenspinors are normalized.

$$\chi_- = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \chi_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The probability of measuring $-\hbar/2$ for the component of spin angular momentum along the z-direction is the modulus squared of the component of $|\chi\rangle$ along $|\chi_-\rangle$.

$$\begin{aligned} P\left(-\frac{\hbar}{2}\right) &= |c_-|^2 \\ &= |\langle\chi_- \mid \chi\rangle|^2 \\ &= \left|\chi_-^\dagger \chi\right|^2 \\ &= \left|\begin{bmatrix} 0 \\ 1 \end{bmatrix}^\dagger \frac{1}{3}\begin{bmatrix} 1-2i \\ 2 \end{bmatrix}\right|^2 \\ &= \frac{1}{9}\left|\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 1-2i \\ 2 \end{bmatrix}\right|^2 \\ &= \frac{1}{9}|0(1-2i)+1(2)|^2 \\ &= \frac{4}{9} \end{aligned}$$

The probability of measuring $+\hbar/2$ for the component of spin angular momentum along the z-direction is the modulus squared of the component of $|\chi\rangle$ along $|\chi_+\rangle$.

$$\begin{aligned} P\left(+\frac{\hbar}{2}\right) &= |c_+|^2 \\ &= |\langle\chi_+ \mid \chi\rangle|^2 \\ &= \left|\chi_+^\dagger \chi\right|^2 \\ &= \left|\begin{bmatrix} 1 \\ 0 \end{bmatrix}^\dagger \frac{1}{3}\begin{bmatrix} 1-2i \\ 2 \end{bmatrix}\right|^2 \\ &= \frac{1}{9}\left|\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1-2i \\ 2 \end{bmatrix}\right|^2 \\ &= \frac{1}{9}|1(1-2i)+0(2)|^2 \\ &= \frac{1}{9}[1^2+(-2)^2] \\ &= \frac{5}{9} \end{aligned}$$

The expectation value of S_z is

$$\langle S_z \rangle = P(\lambda_-)\lambda_- + P(\lambda_+)\lambda_+ = \frac{4}{9}\left(-\frac{\hbar}{2}\right) + \frac{5}{9}\left(\frac{\hbar}{2}\right) = \frac{\hbar}{18}.$$

Part (c)

The matrix representing the x-component of the spin angular momentum S_x is

$$\mathsf{S}_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{\hbar}{2} \\ \frac{\hbar}{2} & 0 \end{bmatrix}.$$

Determine its eigenvalues.

$$\det(\mathsf{S}_x - \lambda \mathsf{I}) = 0$$

$$\begin{vmatrix} 0 - \lambda & \frac{\hbar}{2} \\ \frac{\hbar}{2} & 0 - \lambda \end{vmatrix} = 0$$

$$(0 - \lambda)^2 - \left(\frac{\hbar}{2}\right)\left(\frac{\hbar}{2}\right) = 0$$

$$\lambda^2 - \frac{\hbar^2}{4} = 0$$

$$\lambda = \pm\frac{\hbar}{2}$$

These are the possible values that can be obtained by measuring S_x. Now let

$$\lambda_- = -\frac{\hbar}{2} \quad \text{and} \quad \lambda_+ = +\frac{\hbar}{2},$$

and determine the corresponding eigenvectors (or rather eigenspinors in this context).

$$(\mathsf{S}_x - \lambda_- \mathsf{I})\chi_-^{(x)} = 0 \qquad\qquad (\mathsf{S}_x - \lambda_+ \mathsf{I})\chi_+^{(x)} = 0$$

$$\begin{bmatrix} \frac{\hbar}{2} & \frac{\hbar}{2} \\ \frac{\hbar}{2} & \frac{\hbar}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad\qquad \begin{bmatrix} -\frac{\hbar}{2} & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -\frac{\hbar}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} \frac{\hbar}{2}x_1 + \frac{\hbar}{2}x_2 &= 0 \\ \frac{\hbar}{2}x_1 + \frac{\hbar}{2}x_2 &= 0 \end{aligned}\right\} \qquad\qquad \left.\begin{aligned} -\frac{\hbar}{2}x_1 + \frac{\hbar}{2}x_2 &= 0 \\ \frac{\hbar}{2}x_1 - \frac{\hbar}{2}x_2 &= 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} x_1 + x_2 &= 0 \\ x_1 + x_2 &= 0 \end{aligned}\right\} \qquad\qquad \left.\begin{aligned} -x_1 + x_2 &= 0 \\ x_1 - x_2 &= 0 \end{aligned}\right\}$$

Solve for x_2.

$$x_2 = -x_1 \qquad\qquad x_2 = x_1$$

$$\chi_-^{(x)} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix} \qquad\qquad \chi_+^{(x)} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

Choose x_1 so that each eigenspinor is normalized.

$$|x_1|^2 + |-x_1|^2 = 1 \qquad\qquad |x_1|^2 + |x_1|^2 = 1$$

$$x_1^2 + x_1^2 = 1 \qquad\qquad x_1^2 + x_1^2 = 1$$

$$2x_1^2 = 1 \qquad\qquad 2x_1^2 = 1$$

$$x_1^2 = \frac{1}{2} \qquad\qquad x_1^2 = \frac{1}{2}$$

$$x_1 = \frac{1}{\sqrt{2}} \qquad\qquad x_1 = \frac{1}{\sqrt{2}}$$

$$\chi_-^{(x)} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad\qquad \chi_+^{(x)} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The probability of measuring $-\hbar/2$ for the component of spin angular momentum along the x-direction is the modulus squared of the component of $|\chi\rangle$ along $|\chi_-^{(x)}\rangle$.

$$\begin{aligned}
P\left(-\frac{\hbar}{2}\right) &= \left|c_-^{(x)}\right|^2 \\
&= \left|\langle \chi_-^{(x)} \mid \chi \rangle\right|^2 \\
&= \left|\chi_-^{(x)\dagger} \chi\right|^2 \\
&= \left|\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}^\dagger \frac{1}{3} \begin{bmatrix} 1-2i \\ 2 \end{bmatrix}\right|^2 \\
&= \frac{1}{18} \left|\begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1-2i \\ 2 \end{bmatrix}\right|^2 \\
&= \frac{1}{18} |1(1-2i) + (-1)(2)|^2 \\
&= \frac{1}{18} |-1-2i|^2
\end{aligned}$$

Simplify the result.

$$P\left(-\frac{\hbar}{2}\right) = \frac{1}{18}[(-1)^2 + (-2)^2]$$

$$= \frac{5}{18}$$

The probability of measuring $+\hbar/2$ for the component of spin angular momentum along the x-direction is the modulus squared of the component of $|\chi\rangle$ along $|\chi_+^{(x)}\rangle$.

$$P\left(+\frac{\hbar}{2}\right) = \left|c_+^{(x)}\right|^2$$

$$= \left|\langle\chi_+^{(x)} \mid \chi\rangle\right|^2$$

$$= \left|\chi_+^{(x)\dagger}\chi\right|^2$$

$$= \left|\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}^\dagger \frac{1}{3}\begin{bmatrix}1-2i\\2\end{bmatrix}\right|^2$$

$$= \frac{1}{18}\left|\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1-2i\\2\end{bmatrix}\right|^2$$

$$= \frac{1}{18}|1(1-2i) + 1(2)|^2$$

$$= \frac{1}{18}|3-2i|^2$$

$$= \frac{1}{18}[3^2 + (-2)^2]$$

$$= \frac{13}{18}$$

The expectation value of S_x is

$$\langle S_x\rangle = P(\lambda_-)\lambda_- + P(\lambda_+)\lambda_+ = \frac{5}{18}\left(-\frac{\hbar}{2}\right) + \frac{13}{18}\left(\frac{\hbar}{2}\right) = \frac{2\hbar}{9}.$$

Part (d)

The matrix representing the y-component of the spin angular momentum S_y is

$$\mathsf{S}_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{i\hbar}{2} \\ \frac{i\hbar}{2} & 0 \end{bmatrix}.$$

Determine its eigenvalues.

$$\det(\mathsf{S}_y - \lambda \mathsf{I}) = 0$$

$$\begin{vmatrix} 0-\lambda & -\frac{i\hbar}{2} \\ \frac{i\hbar}{2} & 0-\lambda \end{vmatrix} = 0$$

$$(0-\lambda)^2 - \left(-\frac{i\hbar}{2}\right)\left(\frac{i\hbar}{2}\right) = 0$$

$$\lambda^2 - \frac{\hbar^2}{4} = 0$$

$$\lambda = \pm\frac{\hbar}{2}$$

These are the possible values that can be obtained by measuring S_y. Now let

$$\lambda_- = -\frac{\hbar}{2} \quad \text{and} \quad \lambda_+ = +\frac{\hbar}{2},$$

and determine the corresponding eigenvectors (or rather eigenspinors in this context).

$$(\mathsf{S}_y - \lambda_- \mathsf{I})\chi_-^{(y)} = \mathsf{0} \qquad\qquad (\mathsf{S}_y - \lambda_+ \mathsf{I})\chi_+^{(y)} = \mathsf{0}$$

$$\begin{bmatrix} \frac{\hbar}{2} & -\frac{i\hbar}{2} \\ \frac{i\hbar}{2} & \frac{\hbar}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad\qquad \begin{bmatrix} -\frac{\hbar}{2} & -\frac{i\hbar}{2} \\ \frac{i\hbar}{2} & -\frac{\hbar}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} \frac{\hbar}{2}x_1 - \frac{i\hbar}{2}x_2 = 0 \\ \frac{i\hbar}{2}x_1 + \frac{\hbar}{2}x_2 = 0 \end{aligned}\right\} \qquad\qquad \left.\begin{aligned} -\frac{\hbar}{2}x_1 - \frac{i\hbar}{2}x_2 = 0 \\ \frac{i\hbar}{2}x_1 - \frac{\hbar}{2}x_2 = 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} x_1 - ix_2 = 0 \\ ix_1 + x_2 = 0 \end{aligned}\right\} \qquad\qquad \left.\begin{aligned} -x_1 - ix_2 = 0 \\ ix_1 - x_2 = 0 \end{aligned}\right\}$$

$$x_2 = -ix_1 \qquad\qquad x_2 = ix_1$$

$$\chi_-^{(y)} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -ix_1 \end{bmatrix} \qquad\qquad \chi_+^{(y)} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ ix_1 \end{bmatrix}$$

Choose x_1 so that each eigenspinor is normalized.

$$|x_1|^2 + |-ix_1|^2 = 1 \qquad\qquad |x_1|^2 + |ix_1|^2 = 1$$

$$x_1^2 + x_1^2 = 1 \qquad\qquad x_1^2 + x_1^2 = 1$$

$$2x_1^2 = 1 \qquad\qquad 2x_1^2 = 1$$

$$x_1^2 = \frac{1}{2} \qquad\qquad x_1^2 = \frac{1}{2}$$

$$x_1 = \frac{1}{\sqrt{2}} \qquad\qquad x_1 = \frac{1}{\sqrt{2}}$$

$$\chi_-^{(y)} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix} \qquad\qquad \chi_+^{(y)} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$$

The probability of measuring $-\hbar/2$ for the component of spin angular momentum along the y-direction is the modulus squared of the component of $|\chi\rangle$ along $|\chi_-^{(y)}\rangle$.

$$\begin{aligned}
P\left(-\frac{\hbar}{2}\right) &= \left|c_-^{(y)}\right|^2 \\
&= \left|\langle \chi_-^{(y)} \mid \chi\rangle\right|^2 \\
&= \left|\chi_-^{(y)\dagger}\chi\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}^\dagger \frac{1}{3}\begin{bmatrix} 1-2i \\ 2 \end{bmatrix}\right|^2 \\
&= \frac{1}{18}\left|\begin{bmatrix} 1 & i \end{bmatrix}\begin{bmatrix} 1-2i \\ 2 \end{bmatrix}\right|^2 \\
&= \frac{1}{18}|1(1-2i) + i(2)|^2 \\
&= \frac{1}{18}|1|^2 \\
&= \frac{1}{18}
\end{aligned}$$

The probability of measuring $+\hbar/2$ for the component of spin angular momentum along the y-direction is the modulus squared of the component of $|\chi\rangle$ along $|\chi_+^{(y)}\rangle$.

$$\begin{aligned}
P\left(+\frac{\hbar}{2}\right) &= \left|c_+^{(y)}\right|^2 \\
&= \left|\langle\chi_+^{(y)} \,|\, \chi\rangle\right|^2 \\
&= \left|\chi_+^{(y)\dagger}\chi\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ i\end{bmatrix}^\dagger \frac{1}{3}\begin{bmatrix}1-2i\\ 2\end{bmatrix}\right|^2 \\
&= \frac{1}{18}\left|\begin{bmatrix}1 & -i\end{bmatrix}\begin{bmatrix}1-2i\\ 2\end{bmatrix}\right|^2 \\
&= \frac{1}{18}|1(1-2i)+(-i)(2)|^2 \\
&= \frac{1}{18}|1-4i|^2 \\
&= \frac{1}{18}[1^2+(-4)^2] \\
&= \frac{17}{18}
\end{aligned}$$

The expectation value of S_y is

$$\langle S_y\rangle = P(\lambda_-)\lambda_- + P(\lambda_+)\lambda_+ = \frac{1}{18}\left(-\frac{\hbar}{2}\right) + \frac{17}{18}\left(\frac{\hbar}{2}\right) = \frac{4\hbar}{9}.$$